Vector calculus - Test 5

10 questions | 60 marks | 75 minutes

Reasoning about speed, rest and numerical arc length.

Equivalent correct methods earn the corresponding marks.

1. Consider the parametric curve $x = t^2, y = t^2 + 1, -2 \le t \le 2$. [6 marks]

(a) Eliminate the parameter and state the traced portion. [3 marks]

$y = x + 1$ because both coordinates use t^2.

$0 \le t^2 \le 4$ gives $0 \le x \le 4$.

Answer: $y = x + 1, 0 \le x \le 4$

(b) Describe its orientation using endpoints and any repeated tracing. [3 marks]

At $t = -2, 0, 2$, the positions are $(4, 5), (0, 1), (4, 5)$.

Every point with $0 < x \le 4$ is visited at two opposite parameter values.

Answer: From (4, 5) to (0, 1), then back to (4, 5); each interior point is traced twice.

2. A moves as $(t, 0)$ and B as $(0, 2 - t)$, in metres for $t \ge 0$. [6 marks]

(a) Write squared separation as a function of t. [3 marks]

The separation vector is $(t, t - 2)$.

$D^2 = t^2 + (t - 2)^2 = 2t^2 - 4t + 4$.

Answer: $D^2 = t^2 + (2 - t)^2 = 2t^2 - 4t + 4$.

(b) Find closest separation and its time. [3 marks]

$d(D^2)/dt = 4t - 4 = 0$ gives 1; the second derivative is 4.

$D(1) = \sqrt{2}$ m, at $t = 1$ s.

Answer: $\sqrt{2}$ m at t=1 s.

3. $\mathbf{r} = (t^3 - 3t, t^2)$, $-2 \le t \le 2$. [6 marks]

(a) Find the x-coordinate turning parameters. [3 marks]

$dx/dt = 3t^2 - 3 = 0$ gives $t = -1, 1$.

Its sign is positive, negative, positive across these values, so both are coordinate extrema.

Answer: Horizontal-coordinate extrema at t=-1 and 1.

(b) Find velocity and tangent orientation at those parameters. [3 marks]

$\mathbf{v} = (3t^2 - 3, 2t)$.

At $-1, 1$ it is $(0, -2), (0, 2)$, so the tangents are vertical.

Answer: Velocity at those times is (0, -2) and (0, 2), so tangents are vertical.

4. A quadratic Bézier curve has P 0=(0, 0), P 2=(4, 0), and initial derivative (2, 6). Use the standard Bézier parameter interval $0 \le t \le 1$. [6 marks]

(a) Determine its middle control P 1. [3 marks]

$2(P_1 - P_0) = (2, 6)$ with $P_0 = (0, 0)$ gives $P_1 = (1, 3)$.

Answer: $(1, 3)$

(b) Find B (1/2). [3 marks]

$B(1/2) = \frac{1}{4}(0, 0) + \frac{1}{2}(1, 3) + \frac{1}{4}(4, 0) = (1.5, 1.5)$.

Answer: $(1.5, 1.5)$

5. $\mathbf{r} = (t^3, e^{2t}, \sin 3t)$. [6 marks]

(a) Find the velocity function. [3 marks]

Differentiate each component, retaining chain-rule factors 2 and 3.

$\mathbf{v} = (3t^2, 2e^{2t}, 3\cos 3t)$.

Answer: $\mathbf{v} = (3t^2, 2e^{2t}, 3\cos 3t)$

(b) Find acceleration. [3 marks]

Differentiate velocity componentwise.

$\mathbf{a} = (6t, 4e^{2t}, -9\sin 3t)$.

Answer: Acceleration $(6t, 4e^{2t}, -9\sin 3t)$.

6. $\mathbf{r} = (2\cos t, 2\sin t, 3t)$. [6 marks]

(a) Find velocity and acceleration. [3 marks]

$\mathbf{v} = (-2\sin t, 2\cos t, 3)$ and $\mathbf{a} = (-2\cos t, -2\sin t, 0)$.

Answer: $\mathbf{v} = (-2\sin t, 2\cos t, 3), \mathbf{a} = (-2\cos t, -2\sin t, 0)$

(b) Find speed and explain its constancy. [3 marks]

$|\mathbf{v}|^2 = 4\sin^2 t + 4\cos^2 t + 9 = 13$.

Therefore speed is the constant $\sqrt{13}$.

Answer: The speed is the constant $\sqrt{13}$.

7. For $x = t^3, y = t^2$, t=0. [6 marks]

(a) Find the point and explain why direct substitution into the derivative ratio is inconclusive. [3 marks]

The point is $(0, 0)$, and both coordinate derivatives vanish at 0.

The quotient dy/dx is therefore not determined by direct substitution.

Answer: Point $(0, 0)$; both coordinate derivatives are zero, so the derivative ratio gives 0/0.

(b) Use a limiting direction to find the tangent line. [3 marks]

For nonzero t, $dx/dy = 3t^2/(2t) = 3t/2 \to 0$.

The limiting tangent direction is vertical, so the line is $x = 0$.

Answer: $x = 0$.

8. $\mathbf{r} = (2\cos(t + \pi/2), 2\sin(t + \pi/2))$. [6 marks]

(a) Describe the locus and find its period. [3 marks]

The locus is the radius-2 circle centred at the origin.

The start is $(0, 2)$ and the period is 2π.

Answer: Circle of radius 2; start (0, 2); period 2π.

(b) Find initial velocity and orientation. [3 marks]

$\mathbf{v} = (-2\sin(t + \pi/2), 2\cos(t + \pi/2))$.

$\mathbf{v}(0) = (-2, 0)$, giving counterclockwise motion from the top.

Answer: Initial velocity (-2, 0), counterclockwise.

9. $\mathbf{r} = (t, t^2), 0 \le t \le 1$. [6 marks]

(a) Set up the arc-length integral. [3 marks]

$\mathbf{v} = (1, 2t)$, so speed is $\sqrt{1 + 4t^2}$.

$L = \int_0^1 \sqrt{1 + 4t^2}\, dt$.

Answer: $L = \int_0^1 \sqrt{1 + 4t^2}\, dt$

(b) Use numerical integration to find the curve length to five decimal places. [3 marks]

Numerical integration gives 1.4789428575….

To five decimal places, $L = 1.47894$; this exceeds the chord length $\sqrt{2}$.

Answer: $L \approx 1.47894$

10. A particle has $\mathbf{r} = (t^2, t^3 - t)$ for $-1 \le t \le 1$. [6 marks]

(a) Show that its endpoints coincide but it is not stationary at t=0. [3 marks]

$\mathbf{r}(-1) = \mathbf{r}(1) = (1, 0)$.

$\mathbf{v} = (2t, 3t^2 - 1)$, so $\mathbf{v}(0) = (0, -1)$.

Answer: Endpoints (1, 0); velocity at 0 is (0, -1).

(b) Set up an integral for the full travelled distance. [3 marks]

The speed is $\sqrt{(2t)^2 + (3t^2 - 1)^2}$.

Integrate this nonnegative quantity over $[-1, 1]$ to count the full journey.

Answer: $\int_{-1}^1 \sqrt{4t^2 + (3t^2 - 1)^2}\, dt$.