Vector calculus - Test 6

10 questions | 60 marks | 75 minutes

Joined curves and combined parametric reasoning.

Equivalent correct methods earn the corresponding marks.

1. Consider the parametric curve $x = 1 + 2t, y = 4 - t$, $1 \le t \le 4$. [6 marks]

(a) Eliminate the parameter and state the traced portion. [3 marks]

$t = (x - 1)/2$, hence $y = 4 - (x - 1)/2 = (9 - x)/2$.

The interval gives $3 \le x \le 9$.

Answer: $y = (9 - x)/2$, $3 \le x \le 9$

(b) Describe its orientation using endpoints and any repeated tracing. [3 marks]

The endpoints are $(3, 3)$ and $(9, 0)$.

$dx/dt = 2 > 0$ and $dy/dt = -1 < 0$, so travel is right and down without retracing.

Answer: From (3, 3) to (9, 0), moving right and down.

2. A boat has velocity (3, 0) m/s relative to water, while water flows with velocity (0, 1) m/s. It starts at the origin and the opposite bank is x=30 m. [6 marks]

(a) Find its ground-position function. [3 marks]

Ground velocity is $(3, 0) + (0, 1) = (3, 1)$ m/s.

With zero initial position, $\mathbf{r} = (3t, t)$ m.

Answer: $\mathbf{r} = (3t, t)$ m.

(b) Find arrival time, downstream drift and ground distance. [3 marks]

$3t = 30$ gives 10 s, and y is then 10 m.

The ground speed is $\sqrt{10}$ m/s, so distance is $10\sqrt{10}$ m.

Answer: 10 s; 10 m; $10\sqrt{10}$ m.

3. $\mathbf{r} = (2t - t^2, t^2)$, $0 \le t \le 2$. [6 marks]

(a) Find the rightmost point. [3 marks]

$dx/dt = 2 - 2t = 0$ at 1 and $d^2x/dt^2 = -2$.

The endpoint x values are zero; the maximum point is $\mathbf{r}(1) = (1, 1)$.

Answer: Rightmost point (1, 1) at t=1.

(b) Find velocity and speed there. [3 marks]

$\mathbf{v} = (2 - 2t, 2t)$, so $\mathbf{v}(1) = (0, 2)$.

The speed is 2, so the particle is moving.

Answer: Velocity there (0, 2); speed 2, not zero.

4. A quadratic Bézier curve has endpoints P 0=(0, 0), P 2=(6, 0) and B (1/2)=(2.5, 2). Use the standard Bézier parameter interval $0 \le t \le 1$. [6 marks]

(a) Determine the middle control. [3 marks]

$\frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{1}{4}P_2 = (2.5, 2)$.

Thus $\frac{1}{2}P_1 = (1, 2)$ and $P_1 = (2, 4)$.

Answer: $P_1 = (2, 4)$.

(b) Find its endpoint derivatives. [3 marks]

$B'(0) = 2(2, 4) = (4, 8)$.

$B'(1) = 2[(6, 0) - (2, 4)] = (8, -8)$.

Answer: (4, 8) and (8, -8).

5. $\mathbf{v} = (\cos t, \sin t, 2), \mathbf{r}(0) = (0, 0, 1)$. [6 marks]

(a) Find the position function. [3 marks]

Integration gives $(\sin t + C_1,\ -\cos t + C_2, 2t + C_3)$.

The initial vector gives $C_1 = 0, C_2 = 1, C_3 = 1$.

Answer: $\mathbf{r} = (\sin t, 1 - \cos t, 2t + 1)$

(b) Find displacement from 0 to π. [3 marks]

$\mathbf{r}(\pi) = (0, 2, 2\pi + 1)$ and $\mathbf{r}(0) = (0, 0, 1)$.

Subtraction gives $(0, 2, 2\pi)$.

Answer: Displacement from 0 to π is $(0, 2, 2\pi)$.

6. $\mathbf{r} = (t^2 - 2t, t^3 - 3t)$. [6 marks]

(a) Find velocity and acceleration. [3 marks]

$\mathbf{v} = (2t - 2, 3t^2 - 3)$ and $\mathbf{a} = (2, 6t)$.

Answer: $\mathbf{v} = (2t - 2, 3t^2 - 3), \mathbf{a} = (2, 6t)$

(b) Find rest times and acceleration there. [3 marks]

The first velocity component requires $t = 1$; the second is also zero then.

$\mathbf{a}(1) = (2, 6)$, so rest does not imply zero acceleration.

Answer: Rest at 1; acceleration (2, 6).

7. For $x = 2t, y = t^2 - 4t$, t=2. [6 marks]

(a) Find the point and tangent gradient at $t = 2$. [3 marks]

At 2 the point is $(4, -4)$ and the derivative vector is $(2, 0)$.

The tangent gradient is zero.

Answer: Point (4, -4); gradient 0.

(b) Give the tangent line. [3 marks]

The horizontal line through $(4, -4)$ is $y = -4$.

Answer: $y = -4$.

8. $\mathbf{r} = (5\cos(2t), 5\sin(2t), -t)$. [6 marks]

(a) Describe the helix radius and find the time for one planar turn. Does the full position function have a period? [3 marks]

The planar coordinates trace a radius-5 circle, with angular speed 2.

One planar turn takes π time units, but the changing z coordinate means the full helix has no period.

Answer: Helix of radius 5; one planar turn takes π. The full position has no period.

(b) Find speed and vertical change per turn. [3 marks]

Speed is $\sqrt{10^2 + (-1)^2} = \sqrt{101}$.

Over one planar turn the vertical change is $-\pi$, a drop of π.

Answer: Speed $\sqrt{101}$; vertical change $-\pi$ per turn, a drop of π.

9. $\mathbf{r} = (2\cos t, 2\sin t), 0 \le t \le 4\pi$. [6 marks]

(a) Set up the arc-length integral. [3 marks]

$\mathbf{v} = (-2\sin t, 2\cos t)$, so speed is 2.

$L = \int_0^{4\pi} 2\, dt$.

Answer: $L = \int_0^{4\pi} 2dt$

(b) Evaluate the exact curve length. [3 marks]

$L = 2(4\pi) = 8\pi$.

The parameter interval includes two complete circuits.

Answer: $L = 8\pi$

10. A quadratic Bézier path has controls P 0=(0, 0), P 1=(1, 2), P 2=(3, 2). A cubic begins at Q 0=P 2, with Q 1 unknown. Use the standard Bézier parameter interval $0 \le t \le 1$. [6 marks]

(a) Find the quadratic final derivative. [3 marks]

$B'(1) = 2[(3, 2) - (1, 2)] = (4, 0)$.

Answer: $(4, 0)$.

(b) Find Q 1 so the cubic has the same initial derivative, not merely a parallel tangent. [3 marks]

The cubic initial derivative is $3[Q_1 - (3, 2)]$.

Equate it to $(4, 0)$ to obtain $Q_1 = (3, 2) + (4/3, 0) = (13/3, 2)$.

Answer: $Q_1 = (13/3, 2)$.