Real Polynomials - Practice Test 4

10 questions | 45 marks | 65 minutes

Explain why polynomial methods work, and diagnose claims that need stronger justification.

Equivalent correct methods earn the corresponding marks.

1. Let P be a non-zero cubic and Q a non-zero quadratic. [3 marks]

(a) State the degree of PQ. [1 mark]

The leading-term exponents add.

Answer: 5.

(b) Can $3P - 4Q$ have degree 2? Explain. [2 marks]

Q has no cubic term to cancel the non-zero cubic term of $3P$.

Answer: No; it has degree 3.

2. A student tests two polynomials of degree at most 4 at four distinct inputs and finds equal outputs. [5 marks]

(a) Explain why identity has not yet been proved. [2 marks]

Agreement at four points is compatible with a non-zero quartic difference.

Answer: A non-zero degree-4 difference can have four distinct zeros.

(b) Give an example of such a difference, and state how many distinct matching inputs would suffice. [3 marks]

This difference vanishes at 0,1,2,3 but is not zero everywhere. A degree-at-most-4 difference with five distinct zeros must be zero.

Answer: $x(x-1)(x-2)(x-3)$; five inputs suffice.

3. $P(1) = 2$ and $P(-1) = 4$. [5 marks]

(a) Find its remainder on division by $x^2 - 1$. [3 marks]

$m + c = 2$ and $-m + c = 4$ give $m = -1, c = 3$.

Answer: $R(x) = -x + 3$.

(b) Explain why $P(x) + x - 3$ is divisible by $x^2 - 1$. [2 marks]

$P = (x^2 - 1)Q + (-x + 3)$, so $P + x - 3 = (x^2 - 1)Q$.

Answer: The added expression cancels the remainder.

4. A monic cubic has a double root at 1 and a simple root at -2. [4 marks]

(a) Find its expanded rule. [3 marks]

Expand $(x - 1)^2(x + 2)$.

Answer: $P(x) = x^3 - 3x + 2$.

(b) State the graph behaviour at both roots. [1 mark]

Even multiplicity keeps the sign; odd multiplicity changes it.

Answer: Touches at 1 and crosses at -2.

5. Let $P(x) = x^3 - 4x^2 + 5x - 2$. [5 marks]

(a) Show that 1 has multiplicity at least 2. [3 marks]

Divide by $x - 1$ to get $x^2 - 3x + 2 = (x - 1)(x - 2)$.

Answer: $P = (x - 1)^2(x - 2)$.

(b) Why is its multiplicity exactly 2? [2 marks]

$1 - 2 = -1 \neq 0$.

Answer: The remaining factor is non-zero at 1.

6. Consider $P(x) = x^4 + 4$. [5 marks]

(a) Verify $P = (x^2 - 2x + 2)(x^2 + 2x + 2)$. [3 marks]

Use the difference-of-squares identity with $A = x^2 + 2$, $B = 2x$.

Answer: The product is $(x^2 + 2)^2 - (2x)^2 = x^4 + 4$.

(b) Explain why P has no real zero. [2 marks]

Each discriminant equals $4 - 8 = -4$.

Answer: Both quadratic factors have negative discriminant; also $x^4 + 4 > 0$.

7. $P(x) = (x - 2)(x - 1)(x + 1)$. A student claims that the solutions of $P(x) = 5$ are $2, 1, -1$. [3 marks]

(a) Refute the claim using a substitution. [1 mark]

For example $P(2) = 0 \neq 5$.

Answer: At each claimed input P equals 0, not 5.

(b) Explain which form an equation needs before the zero-product principle applies. [2 marks]

For $P = 5$, solve $P - 5 = 0$; the original factors of P do not directly give the solutions.

Answer: A product must equal zero.

8. A non-zero quartic is claimed to have five distinct real roots. [4 marks]

(a) Explain the contradiction. [2 marks]

Five distinct linear factors would force degree at least 5.

Answer: A degree-4 non-zero polynomial has at most four distinct zeros.

(b) Could a quartic have one real root with multiplicity 4? Give an example. [2 marks]

The single distinct zero 2 is counted four times.

Answer: Yes: $(x-2)^4$.

9. $P_k(x) = x^3 + kx^2 - x - k$. [6 marks]

(a) Factor and find its real roots in terms of k. [3 marks]

Group $x^2(x+k) - (x+k)$.

Answer: $(x+k)(x-1)(x+1)$; roots $-k, 1, -1$.

(b) For which real k are there only two distinct roots? [3 marks]

The moving root $-k$ must coincide with 1 or -1. The two fixed roots remain distinct.

Answer: $k = 1$ or $k = -1$.

10. Two models are $P(x) = x^2$ and $Q(x) = x^2 + (x-1)(x-2)(x-3)$. [5 marks]

(a) Explain why observations at x=1,2,3 cannot distinguish them. [2 marks]

Both models give $1, 4, 9$ respectively.

Answer: The extra product is zero at all three inputs.

(b) Find their predictions at x=4 and explain the lesson for modelling. [3 marks]

The added product is $3 \cdot 2 \cdot 1 = 6$. Agreement on a limited data set does not establish a unique rule without additional assumptions.

Answer: $P(4) = 16$, $Q(4) = 22$.