Real Polynomials - Practice Test 6

10 questions | 58 marks | 75 minutes

Combine coefficient constraints, repeated-root arguments and parameter-dependent real roots.

Equivalent correct methods earn the corresponding marks.

1. $P_a(x) = (x + a)^3 - x^3$. [3 marks]

(a) Expand P and identify its degree when $a \neq 0$. [2 marks]

The cubic terms cancel, while $3a \neq 0$ is the quadratic coefficient.

Answer: $P = 3ax^2 + 3a^2x + a^3$; degree 2.

(b) What happens when $a = 0$? [1 mark]

All three remaining coefficients are zero.

Answer: P is the zero polynomial, with no ordinary degree.

2. Consider the monic cubic with factors $(x - a)(x - 2a)(x + 3a)$. [6 marks]

(a) Expand it in terms of a. [3 marks]

First multiply $(x - a)(x - 2a) = x^2 - 3ax + 2a^2$, then multiply by $x + 3a$.

Answer: $x^3 - 7a^2x + 6a^3$.

(b) Hence find a and all roots if the polynomial equals $x^3 - 28x + 48$. [3 marks]

$a^2 = 4$ and $a^3 = 8$; the second condition fixes the sign.

Answer: $a = 2$; roots $2, 4, -6$.

3. $P(x) = x^3 + ax^2 + bx + 5$ has remainder 3 on division by $x - 1$ and remainder 9 on division by $x + 1$. [6 marks]

(a) Find a and b. [3 marks]

$a + b = -3$ and $a - b = 5$, giving $a = 1, b = -4$.

Answer: $a = 1, b = -4$.

(b) Find the remainder on division by $x^2 - 1$. [3 marks]

For $R = mx + c$, $m + c = 3$, $-m + c = 9$, so $m = -3, c = 6$.

Answer: $-3x + 6$.

4. P is divided by $(x - 1)^2$, leaving remainder $mx + c$. It is known that $P(1) = 3$ and $P'(1) = 4$. [6 marks]

(a) Use the division identity and its derivative to find m and c. [4 marks]

$P = (x - 1)^2 Q + mx + c$. At 1, $m + c = 3$. Differentiating and setting x=1 gives $m = 4$.

Answer: $m = 4, c = -1$; remainder $4x - 1$.

(b) Does this determine the entire polynomial P? [2 marks]

Any multiple of $(x - 1)^2$ may be added without changing either supplied condition.

Answer: No.

5. $P(x) = x^4 + ax^3 + bx^2 + ax + 1$ has zeros 2 and -1. [7 marks]

(a) Find a and b. [4 marks]

$P(2) = 17 + 10a + 4b = 0$ and $P(-1) = 2 - 2a + b = 0$. Substitute $b = 2a - 2$ to get $a = -\frac{1}{2}$.

Answer: $a = -\frac{1}{2}, b = -3$.

(b) Show that all real roots are $2, \frac{1}{2}, -1$, with -1 repeated. [3 marks]

Expand the stated product or divide by the known factors to obtain the remaining quadratic.

Answer: $2P = (x + 1)^2(2x - 1)(x - 2)$.

6. $P_k(x) = x^4 - 2kx^2 + 1$, where k is real. [7 marks]

(a) Solve for x^2 in terms of k. [3 marks]

Use the quadratic formula after setting $u = x^2$.

Answer: $x^2 = k \pm \sqrt{k^2 - 1}$ when $|k| \geq 1$.

(b) State the number of distinct real roots for $k < 1$, $k = 1$ and $k > 1$. [4 marks]

For $-1 < k < 1$ the u-roots are non-real; for $k \leq -1$ they are negative. At k=1, u=1; for k>1, both u-values are positive and distinct.

Answer: Zero, two, four respectively.

7. Consider the equations $x^3 - 3x = c$ for varying real c. [6 marks]

(a) Find the local maximum and minimum of $x^3 - 3x$. [3 marks]

The derivative is $3x^2 - 3$; classify using the derivative sign or second derivative.

Answer: Maximum $(-1, 2)$ and minimum $(1, -2)$.

(b) How many distinct real solutions are there when $|c| < 2$, $|c| = 2$, and $|c| > 2$? [3 marks]

Compare the horizontal line y=c with the three monotonic parts separated by the turning points; tangency at an extreme creates a repeated root.

Answer: Three, two, one respectively.

8. Let $P(x) = (x + 2)(x - 1)^2$ and $Q(x) = 2P(3 - x)$. [5 marks]

(a) Find the factorised form and roots of Q. [3 marks]

Replace every occurrence of x in P by $3 - x$.

Answer: $Q(x) = 2(5 - x)(2 - x)^2$; roots 5 simple and 2 double.

(b) Find its leading coefficient and y-intercept. [2 marks]

The leading factors give $2(-x)(x^2) = -2x^3$; $Q(0) = 2(5)(4) = 40$.

Answer: Leading coefficient -2; y-intercept 40.

9. A sensor calibration is modelled by a polynomial P of degree at most 3. It reports $P(0) = 1$, $P(1) = 2$, $P(2) = 9$, $P(3) = 28$. [6 marks]

(a) Find a polynomial fitting all four values. [4 marks]

Writing $ax^3 + bx^2 + cx + d$ gives $d = 1$, then $a + b + c = 1$, $4a + 2b + c = 4$, $9a + 3b + c = 9$; solve to get $a = 1, b = c = 0$.

Answer: $P(x) = x^3 + 1$.

(b) Explain why no other polynomial of degree at most 3 can fit the same data. [2 marks]

The degree bound forces that difference to be the zero polynomial.

Answer: The difference would have four distinct zeros but degree at most 3.

10. A polynomial P has the form $P(x) = (x - 2)Q(x) + 7$. [6 marks]

(a) Find the remainder of $P(x)^2 - 14P(x)$ on division by $x - 2$. [3 marks]

At x=2, P=7; hence $7^2 - 14(7) = -49$.

Answer: -49.

(b) Prove that $P(x)^2 - 14P(x) + 49$ is divisible by $(x - 2)^2$. [3 marks]

The given division identity makes $P - 7 = (x - 2)Q$; square both sides. This proves divisibility by the squared factor, not merely by one copy.

Answer: It equals $(P(x) - 7)^2 = (x - 2)^2 Q(x)^2$.